Up-to-homotopy structures on vertex algebras

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History

Our aim is to prove of a conjecture of Lian and Zuckerman[1]:

Topological vertex algebras carry a natural G_∞ structure.

- First explicitly stated as Conjecture 2.3 in an article of Kimura, Voronov and Zuckerman[2].
- Related problem solved by Huang and Zhao[3] for a weaker notion of Gerstenhaber-infinity algebra (see Theorem 4.1 of Voronov[4] for details).

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The structure of BRST cohomology

Lian and Zuckerman

- showed that the cohomology of a topological vertex algebra V is a Gerstenhaber algebra — in fact, a Batalin–Vilkovisky algebra,
    - product induced by $x_{(-1)}y$,
    - bracket induced by $(G_{(0)}x)_{(0)}y$;
- posed the problem of lifting this Gerstenhaber algebra structure on the cohomology to a homotopy algebra structure on V itself.

A consequence of our main result is the following:

Theorem (Lian–Zuckerman conjecture)

Any $\mathbb{Z}_{\geq 0}$-graded topological vertex algebra (such that for each conformal weight the fermionic grading is finite) has a canonical G_∞ structure which extends the Gerstenhaber structure on V_0.

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Basic notions in graded algebra

- Graded vector spaces: $W = \bigoplus_{n\in\mathbb{Z}} W^n$
- kth (de)suspension: $(W[k])^n = W^{n+k}$.
- $f\colon W \to W$ graded map: $f\colon W \to W[k], \quad |f(a)| = |f| + |a|$.
- a bilinear operation $\cdot$ is

  commutative if $a \cdot b - \epsilon_{a,b}\, b \cdot a = 0$

  skew-symmetric if $a \cdot b + \epsilon_{a,b}\, b \cdot a = 0$ $\qquad \epsilon_{a,b} = (-1)^{|a|\,|b|}$
- a Lie bracket is a degree 0 skew-symmetric operation satisfying

$$[[a,b],c] + \epsilon_{a,b}\,\epsilon_{a,c}\,[[b,c],a] + \epsilon_{a,c}\,\epsilon_{b,c}\,[[c,a],b] = 0$$

- $f\colon W \to W$ is a derivation with respect to an operation $\cdot$ if

$$f(a \cdot b) = f(a) \cdot b + \epsilon_{f,a}\, a \cdot f(b)\,.$$

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Gerstenhaber and G_∞-algebras

A Gerstenhaber algebra is a graded vector space W with bilinear operations $\cdot$ of degree zero and $[\ ,\ ]$ of degree -1 such that

- $(W, \cdot)$ is a graded commutative associative algebra
- $(W[1], [\ ,\ ])$ is a graded Lie algebra
- $[a, -] : W \to W$ is a derivation with respect to $\cdot$

Definition [Tamarkin–Tsygan]

A G_∞ structure on a graded vector space V is a degree 1 map

$$\gamma : GA \to GA \qquad\qquad (A = V[1]^*)$$

with $\gamma^2 = 0$ which is a derivation with respect to both $\wedge$ and $[\ ,\ ]$.

$$\mathrm{Lie}(A) = \sum_p L^p A = \sum_p \underbrace{[[[\ldots[A, A], \ldots A], A], A]}_{p \text{ terms}},$$

$$GA = \sum_t \bigwedge\nolimits^t (\mathrm{Lie}(A)[-1])$$

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Unpacking this definition

We introduce the length decomposition $GA = \sum G_m A$

$$G_m A = \sum_{p_1+\cdots+p_t=m} G^{p_1,p_2,\ldots,p_t} A$$

$$G_1 A = A[-1] \qquad G_2 A = A[-1] \wedge A[-1] \oplus [A,A][-1]$$

Remarks

A derivation γ on GA is completely specified by

- Either the restriction $\gamma_1 : G_1 A \to GA$
- or its components $\gamma_1^{m+1} : G_1 A \to G_{m+1} A, \quad (m \geq 0)$

Each component $\gamma_1^{m+1} : G_1 A \to G_{m+1} A$ defines a family of maps

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A notion of partial G_∞ structure

Definition

Suppose V and A as above and $1 \leq r \leq \infty$.
A $G(r)$ structure $\gamma^{\leq r}$ on V is a sequence of degree 1 maps

$$\gamma_1^k : G_1 A \to G_k A \qquad (k-1 < r)$$

(and their extensions $\gamma_i^j : G_i A \to G_j A$) which satisfy (R_1^k) for $k-1 < r$.

$$\sum_{j=i}^{k} \gamma_j^k \gamma_i^j = 0 : G_i A \to G_k A. \qquad (\mathrm{R}_i^k)$$

Lemma

- A $G(r)$ structure $\gamma^{\leq r}$ satisfies (R_i^k) whenever $k-i < r$.
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Inductive step: from $G(r)$ to $G(r+1)$ structures

The relation (R_1^{r+1}) may be written

$$\gamma_1^{r+1} d_1 \; + \; d_{r+1}\gamma_1^{r+1} \; = \; \Gamma_{r+1}$$

where $d_k = \gamma_k^k$ satisfies ${d_k}^2 = 0$ by (R_k^k), and

$$\Gamma_k = -\sum_{j=2}^{k-1} \gamma_j^k \gamma_1^j \;:\; G_1 A \to G_k A.$$

For any $G(r)$ structure, the degree two map Γ_{r+1} is defined.

Key Lemma

Suppose $\gamma^{\leq r}$ is a $G(r)$ structure. Then Γ_{r+1} is a cochain map,

$$d_{r+1}\Gamma_{r+1} \quad = \quad \Gamma_{r+1} d_1 .$$

To extend a $G(r)$ to a $G(r+1)$ structure: lift $\Gamma_{r+1} \in \mathrm{Hom}^2(G_1A, G_{r+1}A)$ to $\gamma_1^{r+1} \in \mathrm{Hom}^1(G_1A, G_{r+1}A)$ with $\partial(\gamma_1^{r+1}) = \Gamma_{r+1}$.

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Main Theorem

Let $V = \bigoplus_{\substack{N(s)\leq n\leq N'(s)\\ s\geq 0}} V_s^n$ a bigraded vector space, together with maps

$$m_1\colon V \to V, \qquad m_2,\ m_{1,1}\colon V\otimes V \to V$$

whose duals define a $G(2)$ structure on V, and a square zero map

$$h\colon V \to V$$

such that, for all elements $v \in V_s$,

$$m_1 hv + hm_1 v = sv.$$

Then any extension of the $G(2)$ structure to a G_∞ structure on V_0 has a canonical extension to a G_∞ structure on V.

Sketch of the Proof

Notation: The dual maps $m_1^* \quad h^*$
give square-zero derivations $d \quad \sigma$
on GA of fermionic degrees $1 \quad -1$ such that

$$d\sigma a + \sigma d a = s a \qquad (a \in GA,\ \|a\| = s)$$

Inductive step

Suppose $s > 0$. Given

- a $G(r)$ structure on V
- a $G(r+1)$ structure on $V_{\leq s-1}$
- maps $\gamma_1^{r+1} : G_1 A \to G_{r+1} A$ defined for $\| \ \| = s$ & $| \ | \leq n$
  with $\gamma_1^{r+1} d a + d\gamma_1^{r+1} a = \Gamma_{r+1} a$ for $\|a\| = s$ & $|a| \leq n-1$

Then for $\| \ \| = s$ & $| \ | = n+1$ the inductive formula

$$\gamma_1^{r+1} a := (\Gamma_{r+1} \sigma a - d\gamma_1^{r+1} \sigma a)/s$$

gives $\gamma_1^{r+1} : G_1 A \to G_{r+1} A$ defined for $\| \ \| = s$ & $| \ | \leq n+1$
with $\gamma_1^{r+1} d a + d\gamma_1^{r+1} a = \Gamma_{r+1} a$ for $\|a\| = s$ & $|a| \leq n$.

Unbounded conformal weight

Let

$$
\begin{aligned}
V_{\le s} &= \bigoplus_{i=0}^{s} V_i & \qquad A_{\le s} &= \bigoplus_{i=0}^{s} A_i \\
V_{\le s} &\hookrightarrow V_{\le s+1} & \qquad A_{\le s+1} &\twoheadrightarrow A_{\le s}
\end{aligned}
$$

Apply THEOREM to get G_∞ algebra structures on these finite sums,

$$\gamma^{\le s}_{p_1,\ldots,p_t} : G_1 A_{\le s} \to G^{p_1,\ldots,p_t} A_{\le s},$$

compatible with the maps induced by the projections.

Now take limits to define the G_∞ algebra structure maps for V,

$$G_1 A \cong \lim_s G_1 A_{\le s} \to \lim_s G^{p_1,\ldots,p_t} A_{\le s} \cong G^{p_1,\ldots,p_t} A .$$

Vertex Algebras

A topological vertex algebra is a tuple

$$\Big(\quad V=\bigoplus_{n,s} V_s^n,\ \ \underset{0}{\overset{0}{1}}\ ,\ \underset{0}{\overset{0}{z}},\ \ \underset{0}{\overset{0}{Y(\ \ ,z)}}\ ,\ \underset{1}{\overset{0}{F}},\ \ \underset{2}{\overset{0}{L}}\ ,\ \underset{1}{\overset{1}{Q}},\ \ \underset{2}{\overset{-1}{G}} \quad \Big)$$

where z is a formal variable, $F, L, Q, G \in V$ with the given bidegrees,

$$Y(\ \ ,z) : V \longrightarrow \mathrm{End}(V)[[z^{\pm 1}]], \qquad Y(a,z)=\sum_{k\in\mathbb{Z}} a_{(k)} z^{-k-1}$$

is a 1-1 map, $a_{k+1-s} = a_{(k)} \in \mathrm{End}(V)$ for $a \in V_s$, satisfying the Cauchy–Jacobi–Borcherds relation:

$$\sum_{j\geq 0}\binom{m}{j}(a_{(n+j)}b)_{(m+k-j)} = \sum_{j\geq 0}(-1)^j\binom{n}{j}\big(a_{(n+m-j)}b_{(k+j)} \pm b_{(n+k-j)}a_{(m+j)}\big)$$

and other axioms including $\quad Q_0Q_0 = G_0G_0 = 0, \quad Q_0G_0 + G_0Q_0 = L_0,$

$$F_0 a = na \ \Leftrightarrow\ a \in V^n \qquad\qquad L_0 a = sa \ \Leftrightarrow\ a \in V_s$$