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Random Effect Models at the Observation Level

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Ch9 Random Effect Models at the Observation Level

9.1 Introduction

Assume, given random effect γ,

that y has conditional probability density function

$$f(y|\gamma)$$

Assume that γ has pdf

$$f(\gamma)$$

Then the (marginal) distribution of Y is a mixture distribution with pdf

$$f_Y(y) = \int f(y|\gamma) f(\gamma) d\gamma$$

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Types of model of random effect

We distinguish three type of model of random effect at the observational level:

(i) when an an explicit continuous mixture distribution exists.

(ii) when a continuous mixture is not explicit but approximated using Gaussian quadrature points

(iii) when a 'non-parametric' mixture [effectively a finite mixture] is assumed

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9.2 Fitting explicit continuous mixture distributions

Example Let $y|\gamma \sim PO(\mu\gamma)$

and $\gamma \sim GA(1, \sigma^{1/2})$

then $y \sim NBI(\mu, \sigma)$

See Table 4.2 in Section 4.1 for further examples.

9.3 Fitting non-explicit continuous mixture distributions using Gaussian quadrature

$Y_i \sim D(\mu_i, \sigma_i, \nu_i, \tau_i)$ where D is any distribution

$$
\begin{aligned}
g_1(\mu_i) &= \eta_{1i} = x_{1i}^T \beta_1 + \gamma_i \\
g_2(\sigma_i) &= \eta_{2i} = x_{2i}^T \beta_2 \\
g_3(\nu_i) &= \eta_{3i} = x_{3i}^T \beta_3 \\
g_4(\tau_i) &= \eta_{4i} = x_{4i}^T \beta_4
\end{aligned}
$$

where $\gamma_i \sim N\left(0, \sigma_\gamma^2\right)$ for $i = 1,2,\ldots,n$.

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Matrix form for model

$Y \sim D(\mu, \sigma, \nu, \tau)$ where D is any distribution

$$
\begin{aligned}
g_1(\mu) &= \eta_1 = X_1 \beta_1 + \gamma \\
g_2(\sigma) &= \eta_2 = X_2 \beta_2 \\
g_3(\nu) &= \eta_3 = X_3 \beta_3 \\
g_4(\tau) &= \eta_4 = X_4 \beta_4
\end{aligned}
$$

where $\gamma^T = (\gamma_1, \gamma_2, .., \gamma_n)$ and $\gamma_i \sim N(0, \sigma_\gamma^2)$

Gaussian quadrature

Let $\gamma_i = \sigma_\gamma Z_i$ for $i=1,2,\ldots,n.$

Gaussian quadrature effectively approximates the Continuous N(0,1) distribution for each Z_i by a discrete distribution,

$Z_i = z_k$ with probability π_k for $k=1,2,\ldots,K$..

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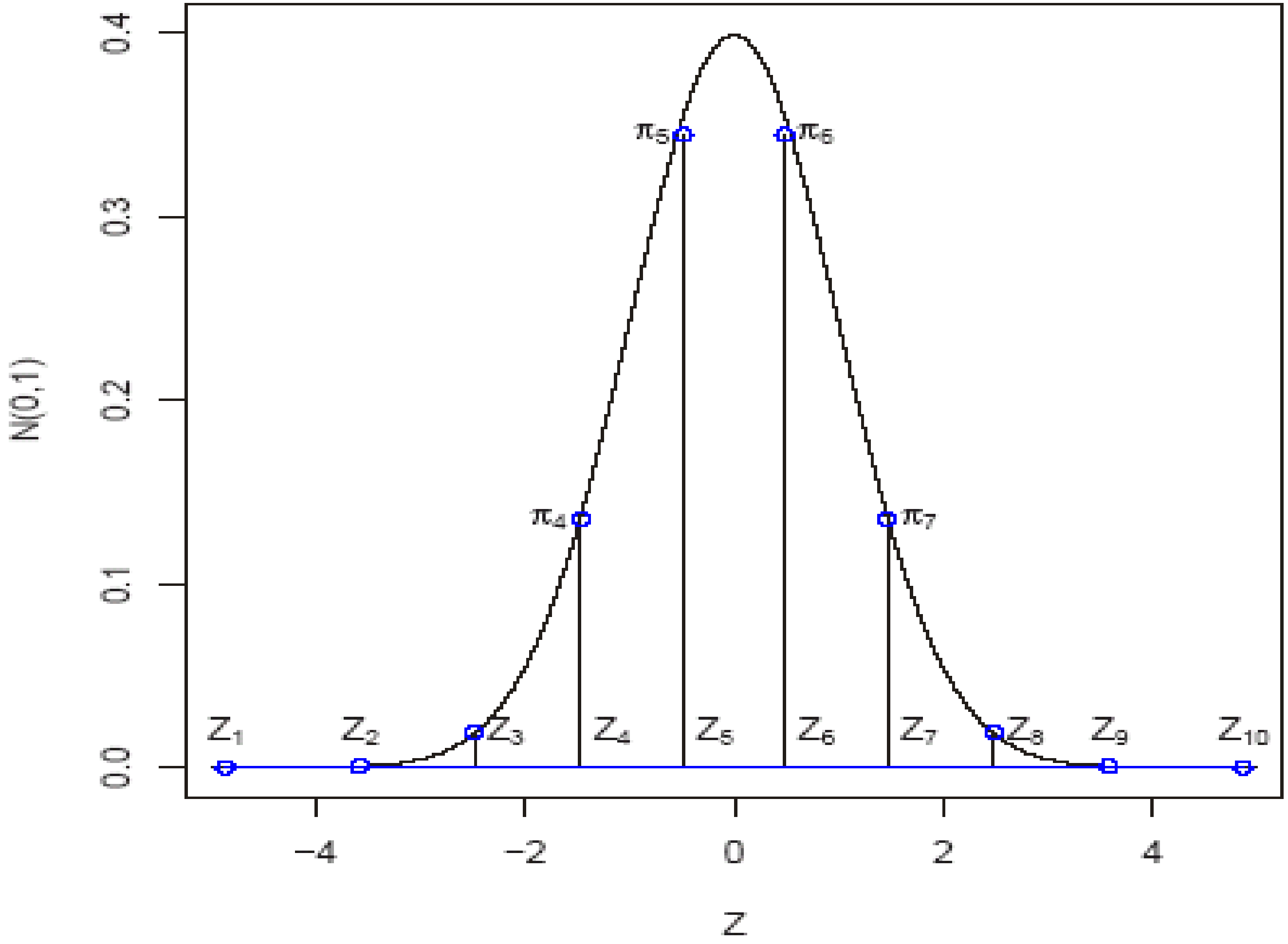
Discrete approximation to N(0,1) using Gaussian quadrature with K=10



Model

$Y \sim D(\mu, \sigma, \nu, \tau)$ where D is any distribution where component k has probability π_k with Predictors:

$$\begin{aligned}
g_1(\mu) &= \eta_1 = X_1\beta_1 + z_k\sigma_\gamma \\
g_2(\sigma) &= \eta_2 = X_2\beta_2 \\
g_3(\nu) &= \eta_3 = X_3\beta_3 \\
g_4(\tau) &= \eta_4 = X_4\beta_4
\end{aligned}$$

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Fitting a normal random effect model

The model can now be considered as a finite mixture of K components in which the π_k's and u_k's are known, and fitted as a mixture model with parameters $\psi = \theta = (\beta_1, \beta_2, \beta_3, \beta_4, \sigma_\gamma)$ in common.

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Summary of iteration (r+1) of the EM algorithm

E-step Replace δ_{ik} by $\hat{w}_{ik}^{(r+1)}$ for all i and k

M-step (1) obtain $\hat{\theta}^{(r+1)}$ by fitting a weighted GAMLSS model to an expanded data set using predictor $\eta_{1e} = X_{1e}\beta_1 + z\sigma_\gamma$ for μ and predictors $\eta_{se} = X_{se}\beta_s$ for s=2,3 and 4 for σ, ν and τ respectively.

Expanded data required for fitting

i	k	y_e	x_{1e}	x_{2e}	x_{3e}	x_{4e}	z	weights $\hat{w}^{(r+1)}$
1	1						z_1	
2	1	y	x_1	x_2	x_3	x_K	z_1	$\hat{w}_1^{(r+1)}$
⋮	⋮						⋮	
n	1						z_1	
1	2						z_2	
2	2	y	x_1	x_2	x_3	x_K	z_2	$\hat{w}_2^{(r+1)}$
⋮	⋮						⋮	
n	2						z_2	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	K						z_K	
2	K	y	x_1	x_2	x_3	x_K	z_K	$\hat{w}_K^{(r+1)}$
⋮	⋮						⋮	
n	K						z_K	

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Model fitting for brain data

br.3 <- gamlssNP(formula = lbrain ~ lbody,
mixture = "gq", K = 20, tol = 1, family = NO)

This model uses a normal random effect for the intercept in the model for μ., fitted using Gaussian quadrature with K=20 quadrature points.

The linear regression model for lbrain against lbody has random intercepts, but the same slope and same σ.

9.4 Fitting non-parametric random effects models

$Y \sim D(\mu, \sigma, \nu, \tau)$ where D is any distribution

$$
\begin{aligned}
g_1(\mu) &= \eta_1 = \mathbf{X}_1\beta_1 + \gamma \\
g_2(\sigma) &= \eta_2 = \mathbf{X}_2\beta_2 \\
g_3(\nu) &= \eta_3 = \mathbf{X}_3\beta_3 \\
g_4(\tau) &= \eta_4 = \mathbf{X}_4\beta_4.
\end{aligned}
$$

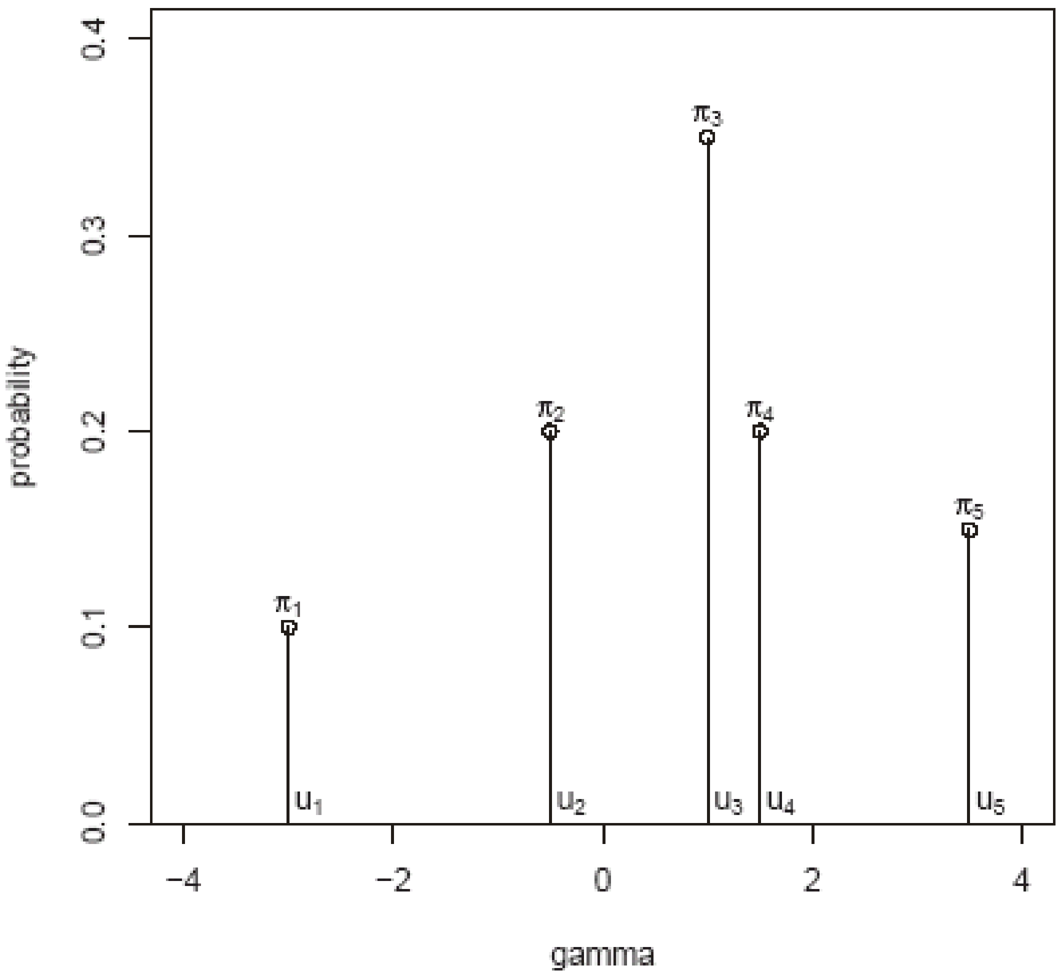
where $\gamma^T = (\gamma_1, \gamma_2, .., \gamma_n)$

‘Non-parametric’ random effect distribution

Let $\gamma_i = u_k$ with probability π_k

for $k=1,2,\ldots,K$ and $i=1,2,\ldots,n$.

‘Non-parametric’ random effect distribution



Model

$Y \sim D(\mu, \sigma, \nu, \tau)$ where D is any distribution

where component k has probability π_k with

Predictors:

$$
\begin{aligned}
g_1(\mu) &= \eta_1 = X_1\beta_1 + \mathbf{1}u_k \\
g_2(\sigma) &= \eta_2 = X_2\beta_2 \\
g_3(\nu) &= \eta_3 = X_3\beta_3 \\
g_4(\tau) &= \eta_4 = X_4\beta_4
\end{aligned}
$$

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Fitting a non-parametric random effect model

The model can now be considered as a finite mixture of K components in which the π_k's and z_k's are unknown, and fitted as a mixture model with parameters $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ in common.

Note that the K components have different μ model intercept parameters $u = (u_1, u_2, \ldots, u_K)$

Summary of iteration (r+1) of the EM algorithm

E-step Replace δ_{ik} by $\hat{w}_{ik}^{(r+1)}$ for all i and k

M-step (1) obtain $\hat{\theta}^{(r+1)}$ by fitting a weighted GAMLSS model to an expanded data set, where $\theta = (\beta, u)$

(2) $\hat{\pi}_k^{(r+1)} = \frac{1}{n}\sum_{i=1}^{n} \hat{w}_{ik}^{(r+1)}$ for $k=1,2,..,K$

(3) $\hat{\psi}^{(r+1)} = \left(\hat{\theta}^{(r+1)}, \hat{\pi}^{(r+1)}\right)$

Expanded data required for fitting

i	MASS	y_3	x_e	$\hat{w}^{(r+1)}$
1	1			
2	1	y	x	$\hat{w}_1^{(r+1)}$
$\vdots$	$\vdots$			
n	1			
1	2			
2	2	y	x	$\hat{w}_2^{(r+1)}$
$\vdots$	$\vdots$			
n	2			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	K			
2	K	y	x	$\hat{w}_K^{(r+1)}$
$\vdots$	$\vdots$			
n	K			

Fitting non-parametric random effects terms

If the factor MASS is included in the predictor for μ,
then the (predictor) intercepts are a non-parametric random term

If an interaction MASS*x is included in the predictor for μ
then (predictor) coefficients in x are a non-parametric random.term.
[The syntax for this for μ in gamlssNP is random = ~ x.]

Similarly for distribution parameters σ, ν and τ.

All other (predictor) parameters are fixed effects terms.

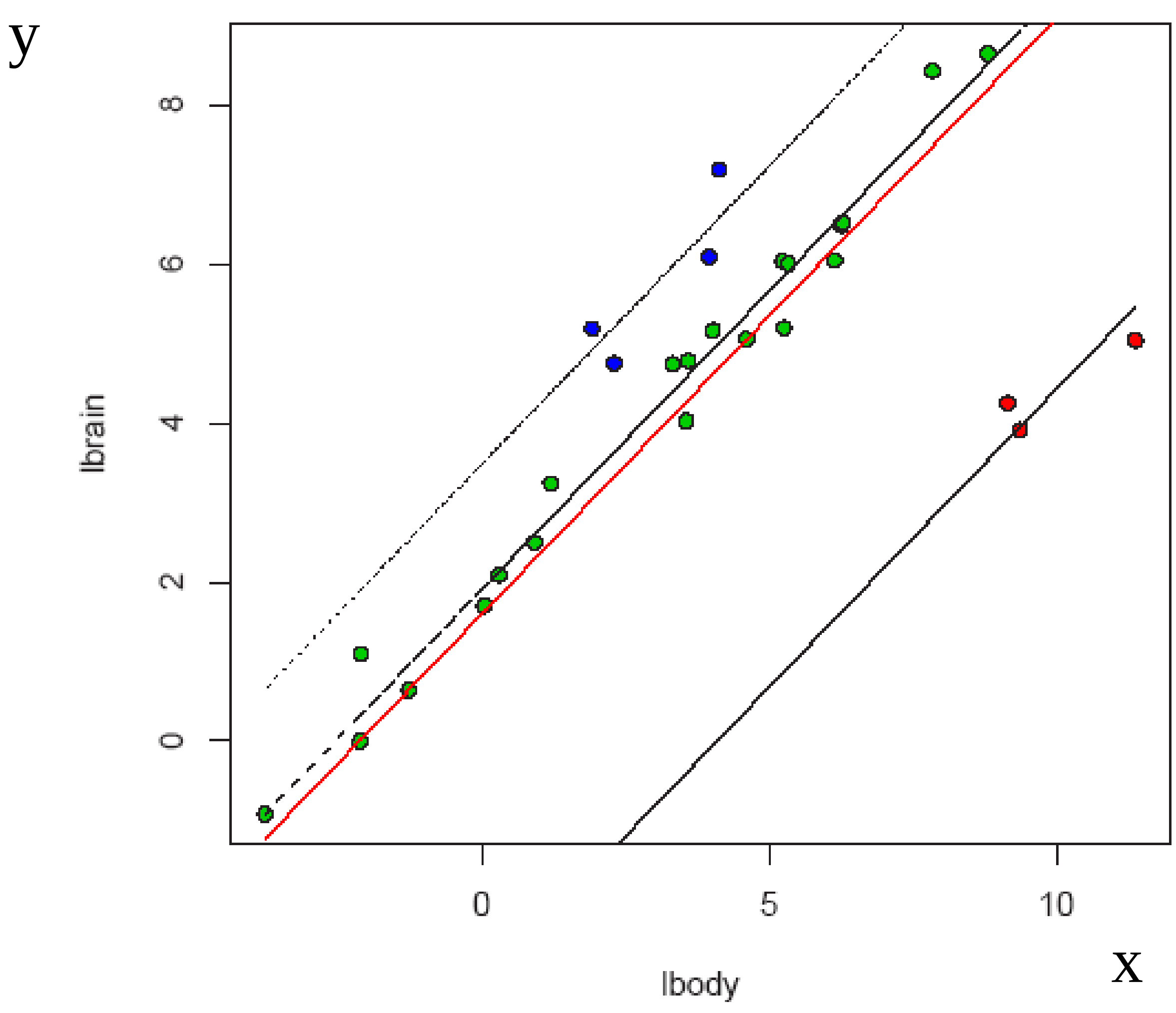
Model fitting for brain data

```
br.3 <- gamlssNP(formula = lbrain ~ lbody,
                 mixture = "np", K = 3, tol = 1, family = NO)
```

This model uses a non-parametric random effect for the intercept in the model for μ., fitted as a finite mixture with 3 components.

The linear regression model for lbrain against lbody has random intercepts, but the same slope and same σ.

Animal brain size data



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Alternative models for brain data

model	μ intercept	μ slope	σ
br.3	different	same	same
br.31	different	same	different
br.32	different	different	same
br.33	different	different	different

In the table ‘different’ is interpreted as ‘non-parametric random’

Non-parametric random effects models

So br.31 has random intercepts for both μ and σ.

br.31 <- gamlssNP(formula = lbrain ~ lbody, sigma.fo = ~MASS,
mixture = "np", K = 3, tol = 1, family = NO)

and br.32 has random intercepts and random slopes for μ.

br.32 <- gamlssNP(formula = lbrain ~ lbody, random = ~lbody,
sigma.fo = ~1, mixture = "np", K = 3, tol = 1, family = NO)

9.5 Conclusion

Use gamlssNP() with mixture="np" for non-parametric random effects. Many non-parametric random effects terms (random intercepts and coefficients in μ, σ, ν and τ) are permitted

Use gamlssNP() with mixture="gq" for a normal random effect, fitted using Gaussian quadrature. Note than only **one** normal randon effect term is permitted. [For more than one, see Chapter 11.]

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